Chapter 1 to 8

/60 marks

1. (a) Show that $2x^2 + x - 15$ can be written in the form $2(x + a)^2 + b$,

where *a* and *b* are **exact** constants to be found.

$2\left(x^2 + \frac{x}{2} - \frac{15}{2}\right)$

[2]

$= 2\left(\left(x + \frac{1}{4}\right)^2 - \frac{1}{16} - \frac{15}{2}\right)$

$= 2\left(\left(x + \frac{1}{4}\right)^2 - \frac{121}{16}\right) \rightarrow 2\left(x + \frac{1}{4}\right)^2 - \frac{121}{8}$

(b) Hence write down the coordinates of the stationary point on the curve $y = 2x^2 + x - 15$.

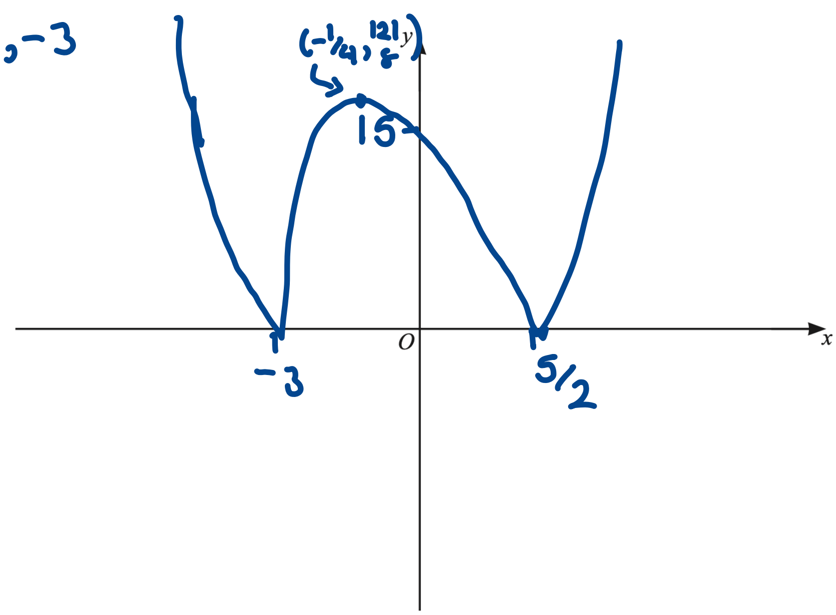
$\left(-\frac{1}{4}, -\frac{121}{8}\right)$

[2]

(c) On the axes, sketch the graph of $y = |2x^2 + x - 15|$, stating the coordinates of the points where the graph meets the coordinate axes.

$y = 0, x = \frac{5}{2}, -3$

$x = 0, y = 15$



[3]

(d) Write down the value of the constant k for which the equation

$|2x^2 + x - 15| = k$ has 3 distinct solutions.

[2]

$k = 121/8$

2. (a)Solve the following simultaneous equations.

$$3y - 2x + 2 = 0$$

$$xy = \frac{1}{2}$$

[3]

$3y - 2x + 2 = 0$

$3y + 2 = 2x$

$x = \frac{3y+2}{2}$

$xy = 1/2$

$\frac{3y+2}{2}(y) = 1/2$

$3y^2 + 2y = 1$

$3y^2 + 2y - 1 = 0$

$(3y-1)(y+1) = 0$

$y = 1/3, -1$

$\therefore x = 3/2, -1/2$

(b) Solve the equation, $log_3 x + 3 = 10\, log_x 3$, giving your answers as powers of 3.

$$\log_3 x + 3 = 10\left(\frac{1}{\log_3 x}\right)$$

[4]

let $\log_3 x = u$

$$u + 3 = \frac{10}{u}$$

$$u^2 + 3u - 10 = 0$$

$$(u-2)(u+5) = 0$$

$$\log_3 x = 2 \text{ or } -5$$

$$x = 3^2 \text{ or } 3^{-5} \rightarrow x = 9 \text{ or } \frac{1}{243}$$

3. A function f(x) is such that $f(x) = e^{3x} - 4$, for $x \in R$.

   a. Find the range of f.

$$f > -4$$

[1]

   b. Find an expression for $f^{-1}(x)$.

$$x = e^{3y} - 4$$

[2]

$$x + 4 = e^{3y}$$

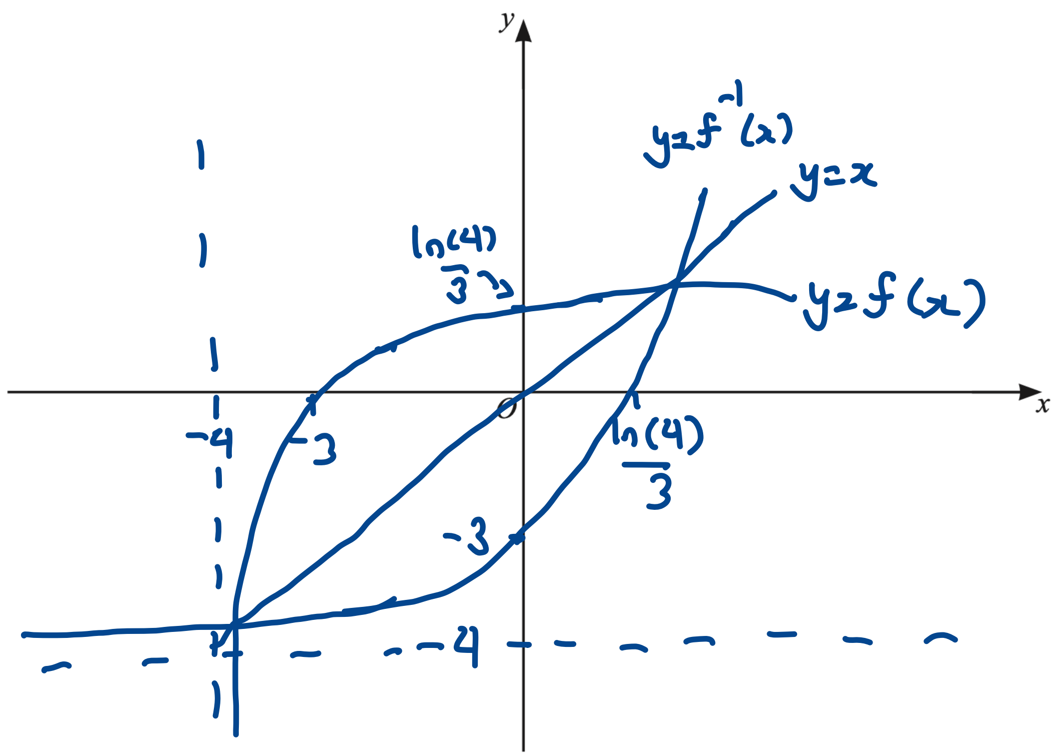
$$\ln(x+4) = 3y$$

$$y = \frac{\ln(x+4)}{3}$$

$$f^{-1}(x) = \frac{\ln(x+4)}{3}$$

c. On the axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ stating the exact values of the intercepts with the coordinate axes.

[4]



$y=e^{3x}-4$

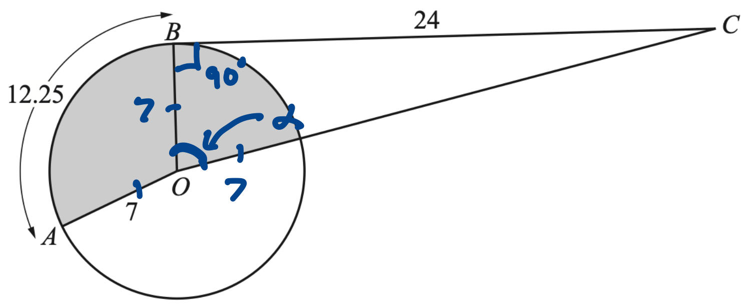
$x=0, y=-3$

$0=e^{3x}-4$

$4=e^{3x}$

$x=\frac{\ln(4)}{3}$

4. In this question all lengths are in metres.



The diagram shows a circle, centre O, radius 7. The points A and B lie on the circumference of the circle. The line BC is a tangent to the circle at the point B such that the length of BC is 24. The length of the minor arc AB is 12.25.

a. Find the obtuse angle AOB, giving your answer in radians.

[2]

arc $= r\theta$

$12.25 = 7\theta$

$\theta = \frac{12.25}{7} = 1.75$ rad

b. Find the perimeter of the shaded region.

[4]

$\tan\alpha = \frac{O}{A}$

$= \frac{24}{7}$

$\alpha = \tan^{-1}(24/7)$

$= 1.287$ rad

Second arc: $r\theta$

$= 7(1.287)$

$= 9.009$

Perimeter $= 12.25 + 9.009 + 7 + 7$

$= 35.259$ cm

c. Find the area of the shaded region.

$$\text{Area} = \tfrac{1}{2}r^2\theta$$

[2]

$$\text{Total Area} = \tfrac{1}{2}(7)^2(1.75) + \tfrac{1}{2}(7)^2(1.287)$$

$$= 74.4\,\text{cm}^2$$

5. The points *P* and *Q* have coordinates (5, -12) and (15, -6) respectively.

The point *R* lies on the line *l*, the perpendicular bisector of the line *PQ*.

The *x*-coordinate of *R* is 7. Find the *y*-coordinate of *R*.

[4]

$$m_{PQ} = \frac{-6 - -12}{15-5} = 3/5 \qquad \text{midpoint}: \left(\frac{5+15}{2}, \frac{-12+-6}{2}\right)$$

$$\text{normal} = \frac{-1}{3/5} = -5/3 \qquad = (10, -9)$$

$$y - -9 = -5/3(x-10)$$

$$y+9 = -5/3x + 50/3$$

$$y = -5/3x + 23/3$$

$$x = 7,\ y = -4$$

$$R \rightarrow (7, -4)$$

6. Solve the following inequality.

$(2x + 3)(x - 4) > (3x + 4)(x - 1)$

[5]

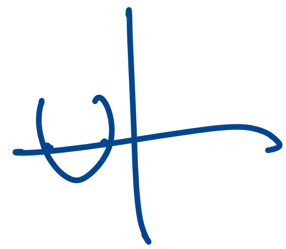
$2x^2 - 8x + 3x - 12 > 3x^2 - 3x + 4x - 4$

$2x^2 - 5x - 12 > 3x^2 - 3x + 4x - 4$

$0 > x^2 + 6x + 8$

$x^2 + 6x + 8 < 0$

$(x + 2)(x + 4) < 0$



$x = -2$ or -4

$-4 < x < -2$

7. Solve the equation $lg(2x - 1) + lg(x + 2) = 2 - lg4$.

[5]

$lg(2x-1)(x+2) = lg100 - lg4$

$(2x-1)(x+2) = 25$

$2x^2 + 4x - x - 2 = 25$

$2x^2 + 3x - 27 = 0$

$(x-3)(2x+9) = 0$

$x = 3$ or $-9/2$ (reject)

$x = 3$

8. The line $y = kx + 6$ intersects the curve $y = x^3 - 4x^2 + 3kx + 2$ at the point where $x = 2$.

(a) Find the value of k. [2]

$2k+6 = 8-16+6k+2$

$12 = 4k$

$k = 3$

(b) Show that, for this value of k, the line cuts the curve only once. [4]

$k=3, \quad y = 3x+6$

$y = x^3 - 4x^2 + 9x + 2$

$3x+6 = x^3-4x^2+9x+2$

$x^3 - 4x^2 + 6x - 4 = 0$

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\
\hline
x-2 & x^3 - 4x^2 + 6x - 4 \\
 & \mp x^3 \pm 2x^2 \\
\hline
 & -2x^2 + 6x \\
 & \mp 2x^2 \pm 4x \\
\hline
 & 2x - 4
\end{array}
$$

$\rightarrow (x-2)(x^2-2x+2)$

x^2-2x+2:

b^2-4ac

$\hookrightarrow (-2)^2 - 4(1)(2)$

$= -4$

$-4 < 0 \therefore$ no other roots

The Maths Society

9. Write $\frac{\sqrt{(9p^2q)}\times r^{-3}}{(2p)^3q^{-1}\sqrt[5]{r}}$ in the form $kp^aq^br^c$, where k, a, b and c are constants.

[4]

$$\frac{(9p^2q)^{1/2}\times r^{-3}}{8p^3q^{-1}r^{1/5}} = \frac{3pq^{1/2}r^{-3}}{8p^3q^{-1}r^{1/5}}$$

$$= \tfrac{3}{8}p^{-2}q^{3/2}r^{-16/5}$$

$k = 3/8$, $a = -2$, $b = 3/2$, $c = -16/5$

10. **DO NOT USE A CALCULATOR IN THIS QUESTION.**

Find the x-coordinates of the points of intersection of the curves

$y = 7x^3 - 7x^2 - 17x - 4$ and $y = x^3 - 2x^2 - 4x - 16.$

[5]

$$7x^3 - 7x^2 - 17x - 4 = x^3 - 2x^2 - 4x - 16$$

$$p(x) = 6x^3 - 5x^2 - 13x + 12 = 0$$

$$p(1) = 6 - 5 - 13 + 12$$
$$= -12 + 12 = 0$$

$$\begin{array}{r|l} & 6x^2 + x - 12 \\ \hline x-1 & 6x^3 - 5x^2 - 13x + 12 \\ & \mp 6x^3 \mp 6x^2 \\ \hline & x^2 - 13x \\ & \mp x^2 \mp x \\ \hline & -12x + 12 \end{array}$$

$$p(x) = (x-1)(6x^2 + x - 12)$$
$$= (x-1)(6x^2 + 9x - 8x - 12)$$
$$= (x-1)(3x(2x+3) - 4(2x+3))$$
$$= (x-1)(3x-4)(2x+3)$$

$x = 1$, $4/3$ or $-3/2$